Dislocated Topologies

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Contents

Slides for presentation of the paper with the same title.

The paper will be submitted to the proceedings.

Motivation: Denotational Semantics

Classical Approach

Given (part of) a computer program P.
Associate operator T_P with it, which is

- monotonic,
- continuous in the Scott topology and
- therefore has a least fixed point M_P by the Knaster-Tarski theorem.

M_P is interpreted as the *denotational semantics* (i.e. the mathematical meaning) of P.

Logic Programming with negation

Associated operators are often not monotonic.
Above mentioned approach is invalid.

Possible solution:
Use alternative fixed-point theorems.

A Generalized Banach Theorem I

X set, $\varrho : X \times X \to \mathbb{R}_0^+$ function. Consider the following conditions for all $x, y, z \in X$.

(Mi) $\varrho(x,x) = 0$
(Mii) if $\varrho(x,y)(= \varrho(y,x)) = 0$ then $x = y$
(Miii) $\varrho(x,y) = \varrho(y,x)$
(Miv) $\varrho(x,y) \leq \varrho(x,z) + \varrho(z,y)$
(Miv') $\varrho(x,y) \leq \max\{\varrho(x,z), \varrho(z,y)\}$

ϱ	(Mi)	(Mii)	(Miii)	(Miv)
metric	x	x	x	x
pseudo-metric	x		x	x
quasi-metric	x	x		x
dislocated metric		x	x	x

Dislocated metric abbreviated as *d-metric*; called *metric domains* by Matthews (1985).

Strong triangle inequality (Miv') holds: prefix *ultra* (e.g. ultrametric, d-ultrametric).

A Generalized Banach Theorem II

Due to Matthews (1985).

We will give alternative proof later.

Definitions on d-metric spaces (X, ϱ)

Convergence: $x_n \to x$ if $\varrho(x_n, x) \to 0$ as $n \to \infty$.

$\smile$ Limits are unique.

Cauchy sequence: (CS)

$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall m, n \geq n_0 : \varrho(x_m, x_n) < \varepsilon.$

$\smile$ Converging sequences are Cauchy sequences.

(X, ϱ) *complete* if all CS converge.

$f : X \to X$ *contraction* if exists $0 \leq \lambda < 1$

s.t. $\varrho(f(x), f(y)) \leq \lambda \varrho(x, y)$ for all x, y.

Generalized Banach Theorem

(X, ϱ) complete d-metric space.

$f : X \to X$ contraction.

Then f has a unique fixed point.

Dislocated Topologies I

We investigate d-metrics
from a topological point of view.

Neighbourhoods

Let X be a set.
Let $\lessdot \subseteq X \times \mathcal{P}(X)$ be a *d-membership relation*
i.e. $x \lessdot A \subseteq B$ implies $x \lessdot B$ for all $A, B \subseteq X$.
For all $x \in X$ let $\emptyset \neq \mathcal{U}_x \subseteq \mathcal{P}(X)$.
$(X, \{\mathcal{U}_x \mid x \in X\}, \lessdot)$ is a *d-topological space*
if for all $x \in X$ we have:

(Ni) $U \in \mathcal{U}_x$ implies $x \lessdot U$.
(Nii) $U, V \in \mathcal{U}_x$ implies $U \cap V \in \mathcal{U}_x$.
(Niii) $\forall U \in \mathcal{U}_x \, \exists V \subseteq U \, \forall y \lessdot V : V \in \mathcal{U}_x \,\&\, U \in \mathcal{U}_y$.
(Niv) $U \in \mathcal{U}_x$ and $U \subseteq V$ imply $V \in \mathcal{U}_x$.

Each $U \in \mathcal{U}_x$ called a *d-neighbourhood* of x.

(X, ϱ) d-metric space.
$B_\varepsilon(x) = \{y \in X \mid \varrho(x, y) < \varepsilon\}$ *balls.*
Define $x \lessdot A$ if exists $\varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq A$.
Define $\mathcal{U}_x = \{A \subseteq X \mid x \lessdot A\}$.
$(X, \{\mathcal{U}_x \mid x \in X\}, \lessdot)$ is a d-topological space.

Dislocated Topologies II

Convergence

$(X, \mathcal{U}, \lessdot)$ d-top. space, $x \in X$, (x_λ) net.
x_λ *d-converges* to $x \in X$ if for all $U \in \mathcal{U}_x$
exists λ_0 s.t. $x_\lambda \in U$ for all $\lambda > \lambda_0$.

$\smile$ Constant sequences may not converge.

$\smile$ Convergence in a d-metric is exactly
d-convergence in underlying d-topology.

Continuity

$(X, \mathcal{U}, \lessdot_X)$, $(Y, \mathcal{V}, \lessdot_Y)$ d-topological spaces.
$f : X \to Y$ is *d-continuous at* $x_0 \in X$ if
for each $V \in \mathcal{V}_{f(x_0)}$ exists $U \in \mathcal{U}_{x_0}$ s.t. $f(U) \subseteq V$.

f *d-continuous* if d-continuous at x for all $x \in X$.

$\smile$ $f : X \to Y$ d-continous iff for each net x_λ
with limit x_0, $f(x_\lambda)$ is a net with limit $f(x_0)$.

Dislocated Metrics I

Continuity

$\smile$ X, Y d-metric spaces, $f : X \to Y$ function.
f d-continuous at $x \in X$ iff for each $\varepsilon > 0$ exists $\delta > 0$ s.t. $f(B_\delta(x)) \subseteq B_\varepsilon(f(x_0))$.

$\smile$ Contractions are d-continuous.

Proof of Generalized Banach Theorem now analogous to proof in metric case.

Obtaining d-Metrics from Metrics

(X, ϱ) d-metric space.
$u_\varrho : X \to \mathbb{R} : x \mapsto \varrho(x, x)$ *dislocation function.*

$\smile$ u_ϱ is d-continuous.

$\smile$ (X, d) metric space, $u : X \to \mathbb{R}_0^+$ function,
$T : \mathbb{R}^2 \to \mathbb{R}_0^+$ symmetric with triangle inequality.
Then $\varrho(x, y) := d(x, y) + T(u(x), u(y))$ d-metric.

$\smile$ Furthermore,
if u, T continuous and $T(x, x) = x$ for all x
then if (X, d) complete, so is (X, ϱ).

Dislocated Metrics II

⌣ (X, d) ultrametric space, $u : X \to \mathbb{R}_0^+$ function.
Then $\varrho(x, y) := \max\{d(x, y), u(x), u(y)\}$ is d-ultrametric and $\varrho(x, x) = u(x)$ for all $x \in X$.

⌣ Furthermore, if u continuous then if (X, d) complete, so is (X, ϱ).

Examples for d-metric spaces.

a) $(\mathbb{R}_0^+, \varrho)$ with $\varrho : (x, y) \to x + y$.

Define $T : \mathbb{R}_0^+ \times \mathbb{R}_0^+ : (x, y) \mapsto \frac{1}{2}(x + y)$.

b) $d(x, y) := \frac{1}{2}|x - y|$ metric on $\mathbb{R}_0^+$, $u = \mathrm{id}_{\mathbb{R}_0^+}$.
Then $\varrho(x, y) := d(x, y) + T(u(x), u(y))$ d-metric with $\varrho(x, y) = \max\{x, y\}$.

c) $\mathcal{I}$ set of all closed intervals on $\mathbb{R}$.
$d([a, b], [c, d]) := \frac{1}{2}(|a - c| + |b - d|)$ metric.
$u([a, b]) := b - a$.
Then $\varrho(I, J) := d(I, J) + T(u(I), u(J))$ d-metric with $\varrho([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$.

Application: Acceptable Programs I

Logic Programs

Logic program: finite set of first-order clauses $\forall(A \leftarrow L_1 \wedge \cdots \wedge L_n)$, written $A \leftarrow L_1, \ldots, L_n$. Atom A *head*, literals $L_1, \ldots, L_n$ *body* of clause.

B_P Herbrand base.
$I_P = \mathcal{P}(B_P)$ set of all interpretations.

Single-step operator $T_P : I_P \to I_P$ defined as $T_P(I)$ is set of all $A \in B_P$ s.t. there exists ground instance $A \leftarrow \texttt{body}$ of clause in P with $I \models \texttt{body}$.

I is model of P iff $T_P(I) \subseteq I$.
T_P not monotonic if negation occurs in P.
Fixed points of T_P are called *supported models.*

Acceptable Programs (Apt & Pedreschi 1993)

P logic program, p, q predicate symbols.
p refers to q if there is clause in P with p in head, q in body.
p depends on q: refl., trans. closure of "refers to".
Neg_P: pred. symbols from neg. literals in P.
Neg_P^*: predicate symbols on which the predicate symbols in Neg_P depend.

Application: Acceptable Programs II

P^-: all clauses with head containing a predicate symbol from Neg_P^*.

P called *acceptable* (wrt. $I \in I_P$, $l : B_P \to \mathbb{N}$) if
- I restricted to Neg_P^* is supported model of P^-,
- for each ground instance $A \leftarrow L_1, \ldots, L_n$ of a clause in P and for all i,
$I \models \bigwedge_{j=1}^{i-1} L_j$ implies $l(A) > l(L_i)$.

Define complete ultrametric on I_P: $J, K \in I_P$,
$d(K, K) = 0$ and $d(J, K) = 2^{-n}$,
where J, K differ on A with $l(A) = n$
but coincide on all B with $l(B) < n$.

$f : I_P \to \mathbb{R} : f(K) = 0$ if $K \subseteq I$ and
$f(K) = 2^{-n}$, where n smallest such that exists
$A \in B_P$ with $l(A) = n$, $K \models A$ and $I \not\models A$.
$u : I_P \to \mathbb{R} : K \mapsto \max\{f(K'), d(K', I)\}$ where
K' is K restricted to pred. symbols not in Neg_P^*.

$\smile$ $\varrho(J, K) := \max\{d(J, K), u(J), u(K)\}$
is complete d-ultrametric on I_P.
$\smile$ T_P is contractive wrt. ϱ.

Conclusions

• Results analogous to conventional topology.

d-uniformities?
d-open sets?

• Do these spaces appear elsewhere in Mathematics or Theoretical Computer Science?

Then further theoretical investigations needed.

• Apply to more general classes of logic programs.

• Interpret $u_\varrho(x)$ as *undesirability*.

Appropriate for settings other than Logic Programming?

• Even more general fixed-point theorems.

Merge with Priess-Crampe & Ribenboim (2000) Theorem?